

# Bound - states for spiked harmonic oscillators and truncated Coulomb potentials

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## Abstract

We propose a new analytical method to solve for the nonexactly solvable Schrödinger equation. Successfully, it is applied to a class of spiked harmonic oscillators and truncated Coulomb potentials. The utility of this method could be extended to study other systems of atomic, molecular and nuclear physics interest.

In atomic, molecular and nuclear physics, spiked harmonic oscillators and truncated Coulomb potentials are of significant interest. Realistic interaction potentials often have a usually repulsive core [1-5]. The simplest model of such a core is provided by the spiked harmonic oscillators

$$V(q) = c_1 q^2 + c_2 q^{-b}, \quad c_1, c_2, b > 0. \quad (1)$$

On the other hand, the truncated Coulomb potential has been founded to be pertinent in the study of the energy levels of the hydrogen - like atoms exposed to intense laser radiation [6-11]. It has been shown [8,10] that under Kramers - Henneberger transformation [12] the laser - dressed binding potential for the hydrogenic system, often called laser - dressed Coulomb potential, may be well simulated by

$$V(q) = -\frac{e^2}{(q^2 + c^2)^{1/2}} ; \quad c > 0. \quad (2)$$

Where the truncation parameter  $c$  is related to the strength of the irradiating laser field.

Thus it is interesting to carry out systematic studies of the bound - states of these potentials. Hall and Saad [4] have studied the spiked harmonic oscillator potentials via smooth transformations method (STM) of the exactly solvable potential  $V(q) = c_2 q^2 + c_2 q^{-2}$  to obtain lower and/or upper energy bounds. They have also calculated the energy eigenvalues using direct numerical integrations of Schrödinger equation [4]. Dutt et al [6] have used a shifted 1/N expansion technique (SLNT) to carry out the energy levels of the laser - dressed Coulomb potential and compared their results also with those of direct numerical integrations [7]. Nevertheless, neither SLNT nor STM is utilitarian in terms of calculating the eigenvalues and eigenfunctions in one batch. Because of the complexity in handling large - order corrections of the standard Rayleigh - Schrödinger perturbation theory, only low - order calculations have been reported for SLNT [6,13] and large - order calculations have been neglected. Eventually, the results of SLNT are not as accurate as sought after.

In this paper we formulate a method for solving Schrödinger equation. In one batch, one should be able to study not only the eigenvalues but also the eigenfunctions. It simply consists of using  $1/\bar{l}$  as a perturbation expansion parameter. Where,  $\bar{l} = l - \beta$ ,  $l$  is a quantum number, and  $\beta$  is a suitable

shift introduced to avoid the trivial case  $l = 0$ . Hence, hereinafter, it should be called pseudoperturbative shifted -  $l$  expansion technique (PSLET).

The construction of our method starts with the time - independent one - dimensional form of Schrödinger equation, in  $\hbar = m = 1$  units,

$$\left[ -\frac{1}{2} \frac{d^2}{dq^2} + \frac{l(l+1)}{2q^2} + V(q) \right] \Psi_{n_r,l}(q) = E_{n_r,l} \Psi_{n_r,l}(q). \quad (3)$$

Where the quantum number  $l$  may specify parity,  $(-1)^{l+1}$ , in one - dimension ( $l=-1$  or  $l=0$ , and  $q \in (-\infty, \infty)$ ) or angular momentum in three - dimensions ( $l=0, 1, \dots$ , and  $q \in (0, \infty)$ ), and  $n_r=0, 1, \dots$  counts the nodal zeros [6,14-17].

To avoid the trivial case  $l=0$ , the quantum number  $l$  is shifted through the relation  $\bar{l} = l - \beta$ . Eq.(3) thus becomes

$$\left\{ -\frac{1}{2} \frac{d^2}{dq^2} + \tilde{V}(q) \right\} \Psi_{n_r,l}(q) = E_{n_r,l} \Psi_{n_r,l}(q), \quad (4)$$

$$\tilde{V}(q) = \frac{\bar{l}^2 + (2\beta + 1)\bar{l} + \beta(\beta + 1)}{2q^2} + \frac{\bar{l}^2}{Q} V(q). \quad (5)$$

Herein, it should be noted that  $Q$  is a constant that scales the potential  $V(q)$  at large -  $l$  limit and is set, for any specific choice of  $l$  and  $n_r$ , equal to  $\bar{l}^2$  at the end of the calculations [13,14]. And,  $\beta$  is to be determined in the sequel.

Our systematic procedure begins with shifting the origin of the coordinate through

$$x = \bar{l}^{1/2}(q - q_o)/q_o, \quad (6)$$

where  $q_o$  is currently an arbitrary point to perform Taylor expansions about, with its particular value to be determined. Expansions about this point,  $x = 0$  (i.e.  $q = q_o$ ), yield

$$\frac{1}{q^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{q_o^2} x^n \bar{l}^{-n/2}, \quad (7)$$

$$V(x(q)) = \sum_{n=0}^{\infty} \left( \frac{d^n V(q_o)}{dq_o^n} \right) \frac{(q_o x)^n}{n!} \bar{l}^{-n/2}. \quad (8)$$

Obviously, the expansions in (7) and (8) center the problem at an arbitrary point  $q_o$  and the derivatives, in effect, contain information not only at  $q_o$  but

also at any point on the axis, in accordance with Taylor's theorem. Also it should be mentioned here that the scaled coordinate, equation (6), has no effect on the energy eigenvalues, which are coordinate - independent. It just facilitates the calculations of both the energy eigenvalues and eigenfunctions. It is also convenient to expand  $E$  as

$$E_{n_r,l} = \sum_{n=-2}^{\infty} E_{n_r,l}^{(n)} \bar{l}^{-n}. \quad (9)$$

Equation (4) thus becomes

$$\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{q_o^2}{\bar{l}} \tilde{V}(x(q)) \right] \Psi_{n_r,l}(x) = \frac{q_o^2}{\bar{l}} E_{n_r,l} \Psi_{n_r,l}(x), \quad (10)$$

with

$$\begin{aligned} \frac{q_o^2}{\bar{l}} \tilde{V}(x(q)) &= q_o^2 \bar{l} \left[ \frac{1}{2q_o^2} + \frac{V(q_o)}{Q} \right] + \bar{l}^{1/2} \left[ -x + \frac{V'(q_o)q_o^3 x}{Q} \right] \\ &+ \left[ \frac{3}{2} x^2 + \frac{V''(q_o)q_o^4 x^2}{2Q} \right] + (2\beta + 1) \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)}{2} x^n \bar{l}^{-n/2} \\ &+ q_o^2 \sum_{n=3}^{\infty} \left[ (-1)^n \frac{(n+1)}{2q_o^2} x^n + \left( \frac{d^n V(q_o)}{dq_o^n} \right) \frac{(q_o x)^n}{n! Q} \right] \bar{l}^{-(n-2)/2} \\ &+ \beta(\beta + 1) \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{2} x^n \bar{l}^{-(n+2)/2} + \frac{(2\beta + 1)}{2}, \end{aligned} \quad (11)$$

where the prime of  $V(q_o)$  denotes derivative with respect to  $q_o$ . Equation (10) is exactly of the type of Schrödinger equation for one - dimensional anharmonic oscillator

$$\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} w^2 x^2 + \varepsilon_o + P(x) \right] X_{n_r}(x) = \lambda_{n_r} X_{n_r}(x), \quad (12)$$

where  $P(x)$  is a perturbation - like term and  $\varepsilon_o$  is a constant. A simple comparison between Eqs.(10), (11) and (12) implies

$$\varepsilon_o = \bar{l} \left[ \frac{1}{2} + \frac{q_o^2 V(q_o)}{Q} \right] + \frac{2\beta + 1}{2} + \frac{\beta(\beta + 1)}{2\bar{l}}, \quad (13)$$

$$\begin{aligned}
\lambda_{n_r} &= \bar{l} \left[ \frac{1}{2} + \frac{q_o^2 V(q_o)}{Q} \right] + \left[ \frac{2\beta + 1}{2} + (n_r + \frac{1}{2})w \right] \\
&+ \frac{1}{\bar{l}} \left[ \frac{\beta(\beta + 1)}{2} + \lambda_{n_r}^{(0)} \right] + \sum_{n=2}^{\infty} \lambda_{n_r}^{(n-1)} \bar{l}^{-n},
\end{aligned} \tag{14}$$

and

$$\lambda_{n_r} = q_o^2 \sum_{n=-2}^{\infty} E_{n_r, l}^{(n)} \bar{l}^{-(n+1)}, \tag{15}$$

Equations (14) and (15) yield

$$E_{n_r, l}^{(-2)} = \frac{1}{2q_o^2} + \frac{V(q_o)}{Q} \tag{16}$$

$$E_{n_r, l}^{(-1)} = \frac{1}{q_o^2} \left[ \frac{2\beta + 1}{2} + (n_r + \frac{1}{2})w \right] \tag{17}$$

$$E_{n_r, l}^{(0)} = \frac{1}{q_o^2} \left[ \frac{\beta(\beta + 1)}{2} + \lambda_{n_r}^{(0)} \right] \tag{18}$$

$$E_{n_r, l}^{(n)} = \lambda_{n_r}^{(n)} / q_o^2 ; \quad n \geq 1. \tag{19}$$

Here  $q_o$  is chosen to minimize  $E_{n_r, l}^{(-2)}$ , i. e.

$$\frac{dE_{n_r, l}^{(-2)}}{dq_o} = 0 \quad \text{and} \quad \frac{d^2 E_{n_r, l}^{(-2)}}{dq_o^2} > 0. \tag{20}$$

Hereby,  $V(q)$  is assumed to be well behaved so that  $E^{(-2)}$  has a minimum  $q_o$  and there are well - defined bound - states. Equation (20) in turn gives, with  $\bar{l} = \sqrt{Q}$ ,

$$l - \beta = \sqrt{q_o^3 V'(q_o)}. \tag{21}$$

Consequently, the second term in Eq.(11) vanishes and the first term adds a constant to the energy eigenvalues. It should be noted that energy term  $\bar{l}^2 E_{n_r, l}^{(-2)}$  has its counterpart in classical mechanics. It corresponds roughly to the energy of a classical particle with angular momentum  $L_z = \bar{l}$  executing circular motion of radius  $q_o$  in the potential  $V(q_o)$ . This term thus identifies

the leading - order approximation, to all eigenvalues, as a classical approximation and the higher - order corrections as quantum fluctuations around the minimum  $q_o$ , organized in inverse powers of  $\bar{l}$ .

The next leading correction to the energy series,  $\bar{E}_{n_r,l}^{(-1)}$ , consists of a constant term and the exact eigenvalues of the unperturbed harmonic oscillator potential  $w^2x^2/2$ . The shifting parameter  $\beta$  is determined by choosing  $\bar{E}_{n_r,l}^{(-1)}=0$ . This choice is physically motivated. It requires not only the agreements between PSLET eigenvalues and the exact known ones for the harmonic oscillator and Coulomb potentials but also between the eigenfunctions as well. Hence

$$\beta = - \left[ \frac{1}{2} + (n_r + \frac{1}{2})w \right], \quad (22)$$

where

$$w = \sqrt{3 + \frac{q_o V''(q_o)}{V'(q_o)}}. \quad (23)$$

Then equation (11) reduces to

$$\frac{q_o^2}{\bar{l}} \tilde{V}(x(q)) = q_o^2 \bar{l} \left[ \frac{1}{2q_o^2} + \frac{V(q_o)}{Q} \right] + \sum_{n=0}^{\infty} v^{(n)}(x) \bar{l}^{-n/2}, \quad (24)$$

where

$$v^{(0)}(x) = \frac{1}{2} w^2 x^2 + \frac{2\beta + 1}{2}, \quad (25)$$

$$v^{(1)}(x) = -(2\beta + 1)x - 2x^3 + \frac{q_o^5 V'''(q_o)}{6Q} x^3, \quad (26)$$

and for  $n \geq 2$

$$\begin{aligned} v^{(n)}(x) &= (-1)^n (2\beta + 1) \frac{(n+1)}{2} x^n + (-1)^n \frac{\beta(\beta+1)}{2} (n-1) x^{(n-2)} \\ &+ \left[ (-1)^n \frac{(n+3)}{2} + \frac{q_o^{(n+4)}}{Q(n+2)!} \frac{d^{n+2} V(q_o)}{dq_o^{n+2}} \right] x^{n+2}. \end{aligned} \quad (27)$$

Equation (10) thus becomes

$$\begin{aligned} \left[ -\frac{1}{2} \frac{d^2}{dx^2} + \sum_{n=0}^{\infty} v^{(n)} \bar{l}^{-n/2} \right] \Psi_{n_r, l}(x) = \\ \left[ \frac{1}{\bar{l}} \left( \frac{\beta(\beta+1)}{2} + \lambda_{n_r}^{(0)} \right) + \sum_{n=2}^{\infty} \lambda_{n_r}^{(n-1)} \bar{l}^{-n} \right] \Psi_{n_r, l}(x). \end{aligned} \quad (28)$$

When setting the nodeless,  $n_r = 0$ , wave functions as

$$\Psi_{0, l}(x(q)) = \exp(U_{0, l}(x)), \quad (29)$$

equation (28) is readily transformed into the following Riccati equation:

$$\begin{aligned} -\frac{1}{2} [U''(x) + U'(x)U'(x)] + \sum_{n=0}^{\infty} v^{(n)}(x) \bar{l}^{-n/2} = & \frac{1}{\bar{l}} \left( \frac{\beta(\beta+1)}{2} + \lambda_0^{(0)} \right) \\ & + \sum_{n=2}^{\infty} \lambda_0^{(n-1)} \bar{l}^{-n}. \end{aligned} \quad (30)$$

Hereinafter, we shall use  $U(x)$  instead of  $U_{0, l}(x)$  for simplicity, and the prime of  $U(x)$  denotes derivative with respect to  $x$ . It is evident that this equation admits solution of the form

$$U'(x) = \sum_{n=0}^{\infty} U^{(n)}(x) \bar{l}^{-n/2} + \sum_{n=0}^{\infty} G^{(n)}(x) \bar{l}^{-(n+1)/2}, \quad (31)$$

where

$$U^{(n)}(x) = \sum_{m=0}^{n+1} D_{m, n} x^{2m-1} \quad ; \quad D_{0, n} = 0, \quad (32)$$

$$G^{(n)}(x) = \sum_{m=0}^{n+1} C_{m, n} x^{2m}. \quad (33)$$

Substituting equations (31) - (33) into equation (30) implies

$$\begin{aligned}
& - \frac{1}{2} \sum_{n=0}^{\infty} \left[ U^{(n)'} \bar{l}^{-n/2} + G^{(n)'} \bar{l}^{-(n+1)/2} \right] \\
& - \frac{1}{2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left[ U^{(n)} U^{(p)} \bar{l}^{-(n+p)/2} + G^{(n)} G^{(p)} \bar{l}^{-(n+p+2)/2} + 2U^{(n)} G^{(p)} \bar{l}^{-(n+p+1)/2} \right] \\
& + \sum_{n=0}^{\infty} v^{(n)} \bar{l}^{-n/2} = \frac{1}{\bar{l}} \left( \frac{\beta(\beta+1)}{2} + \lambda_0^{(0)} \right) + \sum_{n=2}^{\infty} \lambda_0^{(n-1)} \bar{l}^{-n}, \tag{34}
\end{aligned}$$

where primes of  $U^{(n)}(x)$  and  $G^{(n)}(x)$  denote derivatives with respect to  $x$ . Equating the coefficients of the same powers of  $\bar{l}$  and  $x$ , respectively, (of course the other way around would work equally well) one obtains

$$-\frac{1}{2} U^{(0)'} - \frac{1}{2} U^{(0)} U^{(0)} + v^{(0)} = 0, \tag{35}$$

$$U^{(0)'}(x) = D_{1,0} \quad ; \quad D_{1,0} = -w, \tag{36}$$

and integration over  $dx$  yields

$$U^{(0)}(x) = -wx. \tag{37}$$

Similarly,

$$-\frac{1}{2} [U^{(1)'} + G^{(0)'}] - U^{(0)} U^{(1)} - U^{(0)} G^{(0)} + v^{(1)} = 0, \tag{38}$$

$$U^{(1)}(x) = 0, \tag{39}$$

$$G^{(0)}(x) = C_{0,0} + C_{1,0} x^2, \tag{40}$$

$$C_{1,0} = -\frac{B_1}{w}, \tag{41}$$

$$C_{0,0} = \frac{1}{w} (C_{1,0} + 2\beta + 1), \tag{42}$$

$$B_1 = -2 + \frac{q_o^5}{6Q} \frac{d^3 V(q_o)}{dq_o^3}, \tag{43}$$

$$\begin{aligned}
& -\frac{1}{2}[U^{(2)'} + G^{(1)'}] - \frac{1}{2} \sum_{n=0}^2 U^{(n)} U^{(2-n)} - \frac{1}{2} G^{(0)} G^{(0)} \\
& - \sum_{n=0}^1 U^{(n)} G^{(1-n)} + v^{(2)} = \frac{\beta(\beta+1)}{2} + \lambda_0^{(0)}, \tag{44}
\end{aligned}$$

$$U^{(2)}(x) = D_{1,2}x + D_{2,2}x^3, \tag{45}$$

$$G^{(1)}(x) = 0, \tag{46}$$

$$D_{2,2} = \frac{1}{w} \left( \frac{C_{1,0}^2}{2} - B_2 \right) \tag{47}$$

$$D_{1,2} = \frac{1}{w} \left( \frac{3}{2} D_{2,2} + C_{0,0} C_{1,0} - \frac{3}{2} (2\beta + 1) \right), \tag{48}$$

$$B_2 = \frac{5}{2} + \frac{q_o^6}{24Q} \frac{d^4 V(q_o)}{dq_o^4}, \tag{49}$$

$$\lambda_0^{(0)} = -\frac{1}{2} (D_{1,2} + C_{0,0}^2). \tag{50}$$

... and so on. Thus, one can calculate the energy eigenvalue and the eigenfunctions from the knowledge of  $C_{m,n}$  and  $D_{m,n}$  in a hierarchical manner. Nevertheless, the procedure just described is suitable for systematic calculations using software packages (such as MATHEMATICA, MAPLE, or REDUCE) to determine the energy eigenvalue and eigenfunction corrections up to any order of the pseudoperturbation series.

Although the energy series, Eq.(9), could appear divergent, or, at best, asymptotic for small  $\bar{l}$ , one can still calculate the eigenenergies to a very good accuracy by forming the sophisticated Pade' approximation to the energy series. The energy series, Eq.(9), is calculated up to  $E_{0,l}^{(4)}/\bar{l}^4$  by

$$E_{0,l} = \bar{l}^2 E_{0,l}^{(-2)} + E_{0,l}^{(0)} + \dots + E_{0,l}^{(4)}/\bar{l}^4 + O(1/\bar{l}^5), \tag{51}$$

and with the  $P_3^3(1/\bar{l})$  and  $P_3^4(1/\bar{l})$  Pade' approximants it becomes

$$E_{0,l}[3,3] = \bar{l}^2 E_{0,l}^{(-2)} + P_3^3(1/\bar{l}). \tag{52}$$

and

$$E_{0,l}[3,4] = \bar{l}^2 E_{0,l}^{(-2)} + P_3^4(1/\bar{l}). \tag{53}$$

Hereby, an "if" statement is in point. If the energy series, eq.(9), is a Stieltjes series, though it is difficult to prove, then  $E_{0,l}[3,3]$  and  $E_{0,l}[3,4]$  provide upper and lower bounds to the energy [18,19]. Our strategy is therefore clear.

Let us begin with the spiked harmonic oscillators

$$V(q) = \frac{1}{2}(q^2 + aq^{-b}) \quad (54)$$

for which Eq.(22), with  $n_r = 0$ , implies

$$\beta = -\frac{1}{2}(1 + w) \quad ; \quad w = \sqrt{\frac{8q_o + ab(b-2)q_o^{-(b+1)}}{2q_o - abq_o^{-(b+1)}}}. \quad (55)$$

In turn Eq.(21) reads

$$l + \frac{1}{2} \left( 1 + \sqrt{\frac{8q_o + ab(b-2)q_o^{-(b+1)}}{2q_o - abq_o^{-(b+1)}}} \right) = q_o^2 \sqrt{1 - \frac{ab}{2} q_o^{-(b+2)}}. \quad (56)$$

Equation (56) is explicit in  $q_o$  and evidently a closed form solution for  $q_o$  is hard to find, though almost impossible. However, numerical solutions are feasible. Once  $q_o$  is determined the coefficients  $C_{m,n}$  and  $D_{m,n}$  are obtained in a sequel manner. Consequently, the eigenvalues, Eq.(51), and eigenfunctions, Eqs.(31)-(33), are calculated in the same batch for each value of  $a$ ,  $b$ , and  $l$ . In tables 1 and 2 we list PSLET results  $E_P$ , Eq.(51), along with [3,3] and [3,4] Padé approximants, Eqs.(52) and (53) respectively. The results of the smooth transformations method (STM) [4] and direct numerical integration (DNI) [4] are also displayed for comparison purposes.

Our calculated values of the bound - state energies,  $E_P$ , compare well with those from direct numerical integrations [4]. In table 1 the Padé approximants  $E[3,3]$  and  $E[3,4]$  are almost in total agreement with those of Hall and Saad [4] from DNI of the Schrödinger equation. Moreover, it is evident that  $E[3,3]$  and  $E[3,4]$  have provided upper and lower bounds, respectively, to the energy series. However, the same can not be concluded from table 2. Eventually, our computed values of the bound - state energies,  $E_P$ , do not contradict with the upper and/or lower bounds reported by Hall and Saad [4] from the smooth transformations method (STM).

Moreover, our result for  $b = 2$  listed in table 1 is in excellent agreement with the exact one 65.2534584 obtained from Eq.(2) of ref.[4]. On the other

hand, one would rewrite the centrifugal term in (3) plus the potential (54) as  $l'(l'+1)/(2q^2) + q^2/2$ , where  $l' = -1/2 + \sqrt{(l+1/2)^2 + a}$ , and proceed by shifting the irrational quantum number  $l'$  through  $\bar{l} = l' - \beta$ . In this case, one obtains the known exact result  $E_P = (l' + 3/2)$  for the harmonic oscillator  $q^2/2$  from the leading term  $\bar{l}^2 E^{(-2)}$  and the remainder energy corrections are identically zero.

Next, we consider the laser - dressed Coulomb potential

$$V(q) = -\frac{1}{\sqrt{q^2 + c^2}} \quad , \quad c > 0. \quad (57)$$

In this case

$$w = \sqrt{\frac{q_o^2 + 4c^2}{q_o^2 + c^2}}, \quad (58)$$

and

$$l + \frac{1}{2} \left( 1 + \sqrt{\frac{q_o^2 + 4c^2}{q_o^2 + c^2}} \right) = q_o^2 [q_o^2 + c^2]^{-3/4}. \quad (59)$$

Again, we numerically solve for  $q_o$  and proceed exactly as above to calculate the energy eigenvalues and eigenfunctions in the same batch. In tables 3 and 4 we collect the results for the truncation parameter  $c = 1, 5, 10, 50, 100, 200$  based on our approach. The energies  $E_P$ , Eq.(51), compare well with those of Singh et al. [7] from numerical integrations. The Padé approximants  $E[3,3]$  and  $E[3,4]$  are in almost complete accord with those of Singh et al.[7]. However, they do not provide upper and lower bounds to the energy series, Eq.(51). Perhaps, it should be mentioned that the approximate binding potential Eq.(57) is valid for a hydrogen atom in a laser field which corresponds to a truncation parameter  $c$  in the range 20-60 [6]. Higher and lower values of  $c$  have been considered for academic interest only.

Before we conclude some remarks deserve to be mentioned.

For the two problems discussed in this paper, we have shown that it is an easy task to implement PSLET without having to worry about the ranges of couplings and forms of perturbations in the potential involved. In contrast to the textbook Rayleigh - Schrödinger perturbation theory, an easy feasibility of computation of the eigenvalues and eigenfunctions, in one batch, has been

demonstrated, and satisfactory accuracies have been obtained. Moreover, a nice numerical trend of convergence has been achieved. Nevertheless, another suitable criterion for choosing the value of the shift  $\beta$ , reported in Ref. [14], is also feasible. This reference should be consulted for more details.

It is not easy to prove that the energy series Eq.(51) is a Stieltjes series. But, if it is a Stieltjes series, the  $[N, N]$  and  $[N, N + 1]$  Padé approximants provide upper and lower bounds to the energy series. Table 1 bears this out. Moreover, in view of the results listed in tables 1-4 one can confidently conclude that the  $[3,3]$  and  $[3,4]$  Padé approximants to the energy series Eq.(51) can be used to determine the energy eigenvalues to a very satisfactory accuracy.

From the knowledge of  $C_{m,n}$  and  $D_{m,n}$  one can calculate, in the same batch, the wave functions to study electronic transitions and multiphoton emission occurring in atomic systems in the presence of intense laser fields, for example. Such studies already lie beyond the scope of our present methodical proposal.

Finally, the attendant technique PSLET could be applied to Schrödinger equation with rational potentials, such as the nonpolynomial oscillator  $V(q) = q^2 + \lambda q^2 / (1 + gq^2)$ . This type of potential is an interesting model in laser and quantum field theories [20]. The feasibility of PSLET extends also to a class of screened Coulomb potentials, which have relevance in atomic and plasma physics, and to some other models of interest [21-26, and references therein].

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Table 1: 1s - state energies, in  $\hbar = m = 1$  units, of the potential  $V(q) = (q^2 + 1000/q^b)/2$ . Where  $E_P$  represents PSLET results, Eq.(51),  $E_S^{U,L}$  with  $U$  and  $L$  denote upper and lower bounds from STM [4] , and  $E_N$  from DNI [4].  $E[3,4]$  is the [3,4] Padé approximant obtained by replacing the last  $j$  digits of  $E[3,3]$  with the  $j$  digits in parentheses.

$b$	$E_P$	$E[3,3]$ & $(E[3,4])$	$E_S$	$E_N$
0.5	415.88978	415.889786 (86)	$416.30977^U$	415.88979
1.0	190.72330	190.723308 (07)	$190.99213^U$	190.72331
1.5	104.41022	104.410224 (24)	$104.53993^U$	104.41022
1.9	71.06157	71.0615789 (87)	$71.08686^U$	71.06158
2.0	65.25345	65.2534589 (86)	65.25346	65.25346
2.1	60.15200	60.1520114 (11)	$60.12704^L$	60.15201
2.5	44.95547	44.9554855 (50)	$44.83349^L$	44.95549
3.0	33.31675	33.3167621 (18)	$33.07940^L$	33.31676
3.5	26.10884	26.1088462 (48)	$25.76204^L$	26.10885
4.0	21.36950	21.3694640 (14)	$20.91865^L$	21.36964
4.5	18.10194	18.1018377 (10)	$17.55218^L$	18.10183
5.0	15.76134	15.761144 (25)	$15.11758^L$	15.76113
5.5	14.03138	14.03112 (07)	$13.29842^L$	14.03107
6.0	12.71886	12.71879 (61)	$11.90153^L$	12.71862

Table 2: 1s - state energies, in  $\hbar = m = 1$  units, of the potential  $V(q) = (q^2 + a/q^{5/2})/2$ . Where  $E_P$  represents PSLET results, Eq.(51),  $E_S$  denotes the lower bounds from STM [4] , and  $E_N$  from DNI [4].  $E[3,4]$  is the [3,4] Padé approximant obtained by replacing the last  $j$  digits of  $E[3,3]$  with the  $j$  digits in parentheses.

$a$	$E_P$	$E[3,3]$ & $(E[3,4])$	$E_S$	$E_N$
1000	44.95547	44.9554855 (50)	44.83349	44.95549
100	17.54168	17.541911 (899)	17.41900	17.54189
10	7.73423	7.73606 (548)	7.61169	7.73511
5	6.29679	6.29988 (756)	6.17394	6.29647
1	4.32861	4.528 (290)	4.20453	4.31731
0.5	3.85740	3.8308 (289)	3.74611	3.84855
0.05	3.13431	3.1606 (893)	3.10954	3.15243
0.005	3.01445	3.0199 (201)	3.01178	3.01905

Table 3: Bound - state energies, in  $\hbar = m = 1$  units, of the potential  $V(q) = -(q^2 + c^2)^{-1/2}$  for the 1s, 2p, 3d, and 4f states. Where  $E_P$  represents PSLET results Eq.(51),  $E_{SLNT}$  from SLNT [6], and  $E_N$  from DNI [7].  $E[3, 4]$  is the [3,4] Padé approximant obtained by replacing the last  $j$  digits of  $E[3, 3]$  with the  $j$  digits in parentheses.

$c$	State	$-E_P$	$-E[3, 3]$ & $(-E[3, 4])$	$-E_{SLNT}$	$-E_N$
1	1s	0.27412	0.27478 (62)	0.27596	0.27439
	2p	0.113087	0.11296 (303)	0.112826	0.113024
	3d	0.0544357	0.0544371 (82)	0.054442	0.0544362
	4f	0.03106845	0.03106846 (47)	0.031069	0.03106846
5	1s	0.1070836	0.1070813 (10)	0.107396	0.1070814
	2p	0.06819140	0.06818667 (33)	0.068233	0.06818716
	3d	0.04325586	0.04325730 (20)	0.043247	0.04325755
	4f	0.02810534	0.02810520 (25)	0.028101	0.02810524
10	1s	0.06373831	0.06373817 (21)	0.063820	0.0637389
	2p	0.04620043	0.04619903 (00)	0.046228	0.04619904
	3d	0.03315868	0.03315855 (53)	0.033164	0.03315859
	4f	0.02380662	0.02380672 (71)	0.023806	0.02380674

Table 4: Bound - state energies, in  $\hbar = m = 1$  units, of the potential  $V(q) = -(q^2 + c^2)^{-1/2}$  for the 1s, 2p, 3d, and 4f states. Where  $E_P$  represents PSLET results Eq.(51),  $E_{SLNT}$  from SLNT [6], and  $E_N$  from DNI [7].  $E[3, 4]$  is the [3,4] Padé approximant obtained by replacing the last  $j$  digits of  $E[3, 3]$  with the  $j$  digits in parentheses.

$c$	State	$-E_P$	$-E[3, 3]$ & $(-E[3, 4])$	$-E_{SLNT}$	$-E_N$
50	1s	0.01626071	0.01626072 (71)	0.016263	0.01626072
	2p	0.01408837	0.01408837 (37)	0.014090	0.01408838
	3d	0.01215871	0.01215871 (71)	0.012160	0.01215871
	4f	0.01045842	0.01045842 (42)	0.010459	0.01045842
100	1s	0.00862978	0.00862978 (78)	0.008630	0.00862978
	2p	0.00780013	0.00780013 (13)	0.007800	0.00780013
	3d	0.00703519	0.00703519 (19)	0.007035	0.00703519
	4f	0.00633273	0.00633273 (73)	0.006333	0.00633273
200	1s	0.00450285	0.00450285 (85)	0.004503	0.00450286
	2p	0.00419307	0.00419307 (07)	0.004193	0.00419307
	3d	0.00390020	0.00390020 (20)	0.003900	0.00390020
	4f	0.00362385	0.00362385 (85)	0.003624	0.00362385